High-Order Triangular Finite Elements for Electromagnetic Waves in Anisotropic Media

A. KONRAD, MEMBER, IEEE

Abstract—Specialized functionals are introduced for traveling and circulating electromagnetic waves in planar and axisymmetric two-dimensional geometries, without recourse to complex arithmetic. The singularities in the functional for axisymmetric geometries are eliminated by a transformation of the field components. The transformed fields are approximated by high-order interpolation polynomials over triangular regions in the x-y and r-z coordinate planes. A matrix expression assembled from constant element matrices and geometric factors relating to triangle shape, size, and position is obtained which is the discretized equivalent of the original functional. The necessary element matrices have been computed to sixth-order polynomial approximation. The procedure for assembling a global problem is stated. Finally, a matrix equation is generated by minimizing the discretized functional.

I. Introduction

In a recent paper [1], the author presented a general three-component vector variational formulation of electromagnetic field problems and derived functionals for the wave equation in Cartesian and cylindrical coordinates.

In this paper, specialized functionals are introduced for two commonly encountered wave types in loss-free anisotropic media. Finite element matrices are derived and computed for polynomial approximations of orders 1–6. The formulation permits the analysis of traveling waves in waveguides of arbitrary cross section and of circulating waves in resonators of arbitrary longitudinal cross section, including the study of waves in anisotropic media.

The only three-component vector variational formulation of electromagnetic waves in isotropic media, in recent years, is due to English and Young [2]. Their method is, however, restricted to rectangular or circular geometries. With regard to future work with anisotropic media, English and Young predict that the coefficient matrix will be complex. That this is not necessarily so, is evident from the present work.

It is interesting to note that the formulation given here is more closely related to the work of Stone [3], who presented a finite element formulation for the solution of acoustic wave propagation, than to any of the published methods for electromagnetic wave propagation. Stone's formulation is based on Silvester's high-order finite element method for potential calculations and homogeneous waveguide problems [4], [5], as is the method in this paper, although Stone considers acoustic wave propagation. In connection with the dielectric-loaded waveguide problem, Stone comments that "the variational principle must be modified to accommodate the required interface conditions," apparently not realizing that an analogous three-component formulation of electromagnetic wave problems is also possible. Remarkably, the five representative finite element matrices used by Stone [3], [6] are precisely the same as the ones computed here for traveling waves. Two of these five matrices are given in Silvester's work [4], a third one has been computed independently by Csendes [7], [8] and by Daly [9], [10], and the remaining two have been computed by Stone [3], [6]. However, in each of these cases, the matrices are given only up to fourth order, not six as in the present work. Moreover, in this paper, entirely new finite element matrices are derived for circulating wave problems.

It should be noted that as far as electromagnetic field problems are concerned, the finite element method has been applied previously only to the longitudinal electric and magnetic field vector components [7]–[9] and only to problems involving isotropic media. The finite element formulation presented in this paper, therefore, both complements and provides an alternative to existing methods.

II. Mathematical Representation of Traveling and Circulating Electromagnetic Waves

At and above cutoff, a guided wave in the rectangular coordinate system is characterized by the relative phases of its field vector components and by a propagation constant β. The unknowns are the functions H_x, H_y, and H_z which describe the magnetic field components in the x-y plane and the frequency ω at which the wave occurs. Similarly, in the axisymmetric case, the relative phases and the circulation constant m of the guided wave are known but the frequency ω and the functions describing the magnetic field components in the r-z plane are unknown. The constant m is related to the azimuthal periodicity of the wave and therefore must be an integer (including zero).

In this paper, traveling waves of the form

$$\bar{H} = \left[\bar{1}_x H_x(x,y) + \bar{1}_y H_y(x,y) + \bar{1}_z H_z(x,y)\exp\left(\pm j\frac{\pi}{2}\right)\right] \cdot \exp(j\omega t - j\beta z) \quad (1)$$

are considered in media characterized by permeability and permittivity tensors of the form

$$\hat{q} = \begin{bmatrix} q_{xx} & q_{yx} & +jq_{zx} \\ q_{yx} & q_{yy} & +jq_{zy} \\ -jq_{zx} & -jq_{zy} & q_{zz} \end{bmatrix}, \qquad \hat{q} \equiv \hat{\varepsilon} \text{ or } \hat{\mu}. \quad (2)$$

Manuscript received May 25, 1976; revised October 20, 1976. This work was supported by the National Research Council of Canada. This paper is based on the contents of the author's doctoral dissertation at McGill University, Montreal, P.Q., Canada.
The author is with the National Research Council, Ottawa, Ont., Canada.

These tensors are characteristic of transversely magnetized ferromagnetic materials and of plasma [11], [12]. The magnetization may be in any direction but it is restricted to the x-y plane [13].[1]

In direct analogy to traveling waves in Cartesian coordinates, circulating waves in the (r,θ,z) system are mathematically described by

$$\bar{H} = \left[\bar{1}_r H_r(r,z) + \bar{1}_\theta H_\theta(r,z) + \bar{1}_z H_z(r,z) \exp\left(\pm j\frac{\pi}{2}\right)\right] \cdot \exp(j\omega t - jm\theta). \quad (3)$$

The permeability and permittivity tensors are of the same form as their Cartesian counterparts given by (2) but with the subscripts x and y replaced by r and θ, respectively. At the plane $\theta = 0$ the tensors are identical to the Cartesian tensors [14].

III. Specialized Functionals for Traveling and Circulating Waves

The waves described in the previous section must satisfy the wave equation for source-free media

$$\text{curl}\,(\hat{p}\,\text{curl}\,\bar{H}) - \omega^2\hat{\mu}\bar{H} = 0 \quad (4)$$

where $\hat{p} = \hat{\varepsilon}^{-1}$. It can be shown that the vector function $\bar{H} = \bar{H}$ which is a solution of (4) extremizes the functional

$$F(\bar{H}) = \sum_{i=1}^{3} \iiint_\Omega \{p_{i,i}|(\text{curl}\,\bar{H})_i|^2 + 2\,\text{Re}[p_{i+1,i}(\text{curl}\,\bar{H})^*_{i+1}(\text{curl}\,\bar{H})_i] - \omega^2[\mu_{i,i}|H_i|^2 + 2\,\text{Re}(\mu_{i+1,i}H^*_{i+1}H_i)]\}\,dU \quad (5)$$

where the subscript i is cyclic modulo 3 and the asterisk denotes complex conjugate. In addition, $\bar{H}$ satisfies the natural boundary conditions [1], [15] given by

$$\bar{n} \times (\hat{p}\,\text{curl}\,\bar{H}) = 0 \quad (6)$$

over the surface Γ which encloses the volume Ω. $\bar{n}$ is the outward unit normal vector to Γ. The explicit forms of the functional and the natural boundary conditions in Cartesian and cylindrical coordinates are given in [1] and [15]. The specialized form of $F(\bar{H})$ for traveling waves is given by

$$\begin{aligned} F(\bar{H}) = 2\frac{\pi}{\beta}\iint \Bigg[& p_{xx}\left(\frac{\partial H_z}{\partial y} \pm \beta H_y\right)^2 \\ & + p_{yy}\left(\frac{\partial H_z}{\partial x} \pm \beta H_x\right)^2 + p_{zz}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)^2 \\ & - 2p_{yx}\left(\frac{\partial H_z}{\partial y} \pm \beta H_y\right)\left(\frac{\partial H_z}{\partial x} \pm \beta H_x\right) \\ & \pm 2p_{zx}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\left(\frac{\partial H_z}{\partial y} \pm \beta H_y\right) \\ & \pm 2p_{zy}\left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}\right)\left(\frac{\partial H_z}{\partial x} \pm \beta H_x\right) \\ & - \omega^2(\mu_{xx}H_x^2 + \mu_{yy}H_y^2 + \mu_{zz}H_z^2 + 2\mu_{yx}H_xH_y \\ & \mp 2\mu_{zx}H_xH_z \mp 2\mu_{zy}H_yH_z)\Bigg]\,dx\,dy. \end{aligned} \quad (7)$$

[1] Other magnetization directions give rise to other material property tensor forms and phase relationships between the field vector components. It is beyond the scope of this paper to consider all possible combinations.

The constant $2\pi/\beta$ arises from the integration with respect to z between the beginning and the end of a period in the direction of travel.

For circulating waves the functional is given by

$$\begin{aligned} F(\bar{H}) = 2\pi \iint \Bigg\{ & p_{rr}\left(\sqrt{r}\frac{\partial H_\theta}{\partial z} \mp \frac{mH_z}{\sqrt{r}}\right)^2 + p_{\theta\theta}r\left[\left(\frac{\partial H_r}{\partial z}\right)^2 \right.\\ & \left. + \left(\frac{\partial H_z}{\partial r}\right)^2\right] + p_{zz}\left[\left(\sqrt{r}\frac{\partial H_\theta}{\partial r} + \frac{H_\theta}{\sqrt{r}}\right)^2 \right.\\ & \left. + \left(\frac{mH_r}{\sqrt{r}}\right)^2\right] + 2p_{\theta r}\frac{\partial H_r}{\partial z}\left(\pm mH_z - r\frac{\partial H_\theta}{\partial z}\right) \\ & + 2p_{zr}mH_r\left(\frac{\partial H_\theta}{\partial z} \mp \frac{mH_z}{r}\right) \\ & + 2p_{z\theta}\left[\mp\frac{\partial H_z}{\partial r}\left(H_\theta + r\frac{\partial H_\theta}{\partial r}\right) - mH_r\frac{\partial H_r}{\partial z}\right] \\ & - \omega^2 r(\mu_{rr}H_r^2 + \mu_{\theta\theta}H_\theta^2 + \mu_{zz}H_z^2 + 2\mu_{\theta r}H_rH_\theta \\ & \mp 2\mu_{zr}H_rH_z \mp 2\mu_{z\theta}H_\theta H_z)\Bigg\}\,dr\,dz, \\ & m = 0, \pm 1, \pm 2, \pm 3, \cdots. \end{aligned} \quad (8)$$

The $\pm$ and $\mp$ terms arise because the H_z vector component may either lead or lag the other two components by a time phase of $\pi/2$ radians. Notice that all quantities in (7) and (8) are real. When $m = 0$, only those fields for which $H_r = H_z = 0$ are of practical importance, and this case has been treated elsewhere [16]–[18].

In order to avoid difficulties in trying to integrate the singular terms in the functional (8), the vector function $\bar{H}(r,z)$ will be transformed according to [1], [15]–[18]

$$\bar{H} = \sqrt{r}\,\bar{h}(r,z). \quad (9)$$

The transformation is justified because the field vector components must vanish at $r = 0$. The functional (8) now takes the following form:

$$\begin{aligned} F(\bar{h}) = 2\pi \iint \Bigg\{ & p_{rr}\left(r\frac{\partial h_\theta}{\partial z} \mp mh_z\right)^2 + p_{\theta\theta}\left[\left(r\frac{\partial h_r}{\partial z}\right)^2 \right.\\ & \left. + \left(r\frac{\partial h_z}{\partial r}\right)^2 + rh_z\frac{\partial h_z}{\partial r} + \frac{1}{4}h_z^2\right] \\ & + p_{zz}\left[\left(r\frac{\partial h_\theta}{\partial r} + \frac{3}{2}h_\theta\right)^2 + m^2h_r^2\right] \\ & + 2p_{\theta r}r\frac{\partial h_r}{\partial z}\left(\pm mh_z - r\frac{\partial h_\theta}{\partial z}\right) \\ & + 2p_{zr}mh_r\left(r\frac{\partial h_\theta}{\partial z} \mp mh_z\right) \\ & + 2p_{z\theta}\left[\mp\left(\frac{1}{2}h_z + r\frac{\partial h_z}{\partial r}\right) \right.\\ & \left. \cdot\left(\frac{3}{2}h_\theta + r\frac{\partial h_\theta}{\partial r}\right) - mrh_r\frac{\partial h_r}{\partial z}\right] \\ & - \omega^2 r^2(\mu_{rr}h_r^2 + \mu_{\theta\theta}h_\theta^2 + \mu_{zz}h_z^2 + 2\mu_{\theta r}h_rh_\theta \\ & \mp 2\mu_{zr}h_rh_z \mp 2\mu_{z\theta}h_\theta h_z)\Bigg\}\,dr\,dz, \\ & m = \pm 1, \pm 2, \pm 3, \cdots. \end{aligned} \quad (10)$$

TABLE I

RESULTS OF THE DISCRETIZATION OF REPRESENTATIVE INTEGRALS FOR FUNCTIONAL (10)

Representative integrals for functional (3.7)	Discretized equivalents in matrix form	Expressions for the square matrices containing the geometrical information about the triangular finite element	Expressions for the (m,k)-th entry of the constant finite element matrices $[Y_{ij}]$, $[U_{ij}]$, $[\mathcal{U}_{ij}]$, $[Q_{ij\ell}]$ and $[\mathcal{Q}_{ij\ell}]$
$\iint h_1 h_2 \, drdz$	$[V_1]^t[R][V_2]$	$[R]=2\|A\| \sum_{i=1}^{3} \sum_{j=1}^{3} [Y_{ij}]$	$Y_{ij}^{(m,k)}=\frac{1}{2\|A\|}\iint_{\Delta} \zeta_i \zeta_j \alpha_m \alpha_k \, ds$
$\iint r^2 h_1 h_2 \, drdz$	$[V_1]^t[P][V_2]$	$[P]=2\|A\| \sum_{i=1}^{3} r_i \sum_{j=1}^{3} r_j [Y_{ij}]$	
$\iint r h_1 \frac{\partial h_2}{\partial z} \, drdz$	$[V_1]^t[J][V_2]$	$[J]=\frac{\|A\|}{2A} \sum_{i=1}^{3} r_i \sum_{j=1}^{3} (r_{j+2}-r_{j+1})\{[U_{ij}]+[\mathcal{U}_{ij}]\}$	$U_{ij}^{(m,k)}=\frac{1}{2\|A\|}\iint_{\Delta} \zeta_i (\alpha_k \frac{\partial \alpha_m}{\partial \zeta_j} + \alpha_m \frac{\partial \alpha_k}{\partial \zeta_j}) \, ds$
$\iint r h_1 \frac{\partial h_2}{\partial r} \, drdz$	$[V_1]^t[N][V_2]$	$[N]=\frac{\|A\|}{2A} \sum_{i=1}^{3} r_i \sum_{j=1}^{3} (z_{j+1}-z_{j+2})\{[U_{ij}]+[\mathcal{U}_{ij}]\}$	$\mathcal{U}_{ij}^{(m,k)}=\frac{1}{2\|A\|}\iint_{\Delta} \zeta_i (\alpha_k \frac{\partial \alpha_m}{\partial \zeta_j} - \alpha_m \frac{\partial \alpha_k}{\partial \zeta_j}) \, ds$
$\iint r^2 \frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial z} \, drdz$	$[V_1]^t[D][V_2]$	$[D]=\frac{1}{2\|A\|} \sum_{i=1}^{3} r_i \sum_{j=1}^{3} r_j \sum_{\ell=1}^{3} (r_\ell - r_{\ell+2})(r_\ell - r_{\ell+1})[Q_{ij\ell}]$	$Q_{ij\ell}^{(m,k)}=\frac{1}{2\|A\|}\iint_{\Delta} \zeta_i \zeta_j (\frac{\partial \alpha_m}{\partial \zeta_{\ell+1}} - \frac{\partial \alpha_m}{\partial \zeta_{\ell+2}})(\frac{\partial \alpha_k}{\partial \zeta_{\ell+1}} - \frac{\partial \alpha_k}{\partial \zeta_{\ell+2}}) \, ds$
$\iint r^2 \frac{\partial h_1}{\partial r} \frac{\partial h_2}{\partial r} \, drdz$	$[V_1]^t[E][V_2]$	$[E]=\frac{1}{2\|A\|} \sum_{i=1}^{3} r_i \sum_{j=1}^{3} r_j \sum_{\ell=1}^{3} (z_\ell - z_{\ell+2})(z_\ell - z_{\ell+1})[Q_{ij\ell}]$	
$\iint r^2 \frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial r} \, drdz$	$[V_1]^t[Z][V_2]$	$[Z]=\frac{1}{4\|A\|} \sum_{i=1}^{3} r_i \sum_{j=1}^{3} r_j \sum_{\ell=1}^{3} \{d_\ell [Q_{ij\ell}] + t_\ell [\mathcal{Q}_{ij\ell}]\}$	$\mathcal{Q}_{ij\ell}^{(m,k)}=\frac{1}{2\|A\|}\iint_{\Delta} \zeta_i \zeta_j (\frac{\partial \alpha_m}{\partial \zeta_{\ell+1}} \frac{\partial \alpha_k}{\partial \zeta_\ell} - \frac{\partial \alpha_m}{\partial \zeta_\ell} \frac{\partial \alpha_k}{\partial \zeta_{\ell+1}}) \, ds$
This last integral does not appear in (3.7); However, it would appear in a functional for circulating waves of the type $\bar{1}_r H_r + \bar{1}_\theta j H_\theta + \bar{1}_z H_z$ $(j=\sqrt{-1})$		$d_\ell=(r_{\ell+1}-r_\ell)(z_\ell-z_{\ell+2})+(r_\ell-r_{\ell+2})(z_{\ell+1}-z_\ell)$ $t_\ell=(z_{\ell+1}-z_{\ell+2})(r_\ell-r_{\ell+2})-(z_{\ell+2}-z_\ell)(r_{\ell+2}-r_{\ell+1})$	

NOTE: a) superscript t denotes transposition;
b) subscripts i, j and ℓ are cyclic modulo 3;
c) the range of m and k depends on the degree of the interpolation polynomials α;
d) $[Y_{ij}]$, $[U_{ij}]$, $[Q_{ij\ell}]$ are symmetric and $[\mathcal{U}_{ij}]$, $[\mathcal{Q}_{ij\ell}]$ are antisymmetric matrices;
e) r_i and z_i are triangle vertex coordinates and A is the triangle area.

IV. DISCRETIZATION

The objective is to discretize the functionals (7) and (10) so that they can be written in matrix form. In the process, the vector field is approximated by interpolation polynomials over a general triangular region in such a way that the integrations are performed only once and geometrical information is added only when a specific problem is solved [4]–[9], [15]–[19]. Thus, for a general triangle, the vector function $\bar{h}(r,z)$ [or $\bar{H}(x,y)$] is approximated by a linear combination of a complete set of interpolation polynomials $\{\alpha_i; i = 1,2,3,\cdots,n\}$ each of degree N [4]

$$\bar{h} = \sum_{i=1}^{n} \bar{V}_i \alpha_i(\zeta_1,\zeta_2,\zeta_3), \qquad n = (N+1)(N+2)/2. \tag{11}$$

The coefficients $\bar{V}_i$ represent the values of $\bar{h}$ (or $\bar{H}$) at the interpolation nodes. The polynomials α_i given in [4] have been used extensively in recent years to generate high-order triangular finite elements [5]–[8], [15]–[18]. ζ_1, ζ_2, and ζ_3 are triangle area coordinates [4], [15], [19].

When the expanded integrands of (7) and (10) are examined, one finds that many of the terms are similar in form and that the various terms can be classified into representative groups as shown in the first columns of Tables I and II. The subscripts 1 and 2 in the expressions signify that there may be two distinct field vector components involved. Although the discretization of the representative integrals involves the straightforward substitution of algebraic expressions, the procedure is lengthy and tedious. The discretized equivalents of the functionals can be constructed from the results given in Tables I and II.

The discretized equivalent of (7) is given by

$$\begin{aligned} F(\bar{V}) = {} & 2(\pi/\beta)\{[V_x]^t(p_{zz}[\boldsymbol{D}] + \beta^2 p_{yy}[\boldsymbol{R}])[V_x] \\ & -2[V_y]^t(\beta p_{zx}[\boldsymbol{J}] + \beta^2 p_{yx}[\boldsymbol{R}] + p_{zz}[\boldsymbol{Z}]^t \\ & + \beta p_{zy}[\boldsymbol{N}]^t)[V_x] + [V_y]^t(p_{zz}[\boldsymbol{E}] + 2\beta p_{zx}[\boldsymbol{N}] \\ & + \beta^2 p_{xx}[\boldsymbol{R}])[V_y] \mp 2[V_z]^t(\beta p_{yx}[\boldsymbol{J}]^t + p_{zx}[\boldsymbol{D}] \\ & - \beta p_{yy}[\boldsymbol{N}]^t - p_{zy}[\boldsymbol{Z}]^t)[V_x] \mp 2[V_z]^t(\beta p_{yx}[\boldsymbol{N}]^t \\ & + p_{zy}[\boldsymbol{E}] - \beta p_{xx}[\boldsymbol{J}]^t - p_{zx}[\boldsymbol{Z}])[V_y] \\ & + [V_z]^t(p_{xx}[\boldsymbol{D}] + p_{yy}[\boldsymbol{E}] - 2p_{yx}[\boldsymbol{Z}])[V_z] \\ & - \omega^2(\mu_{xx}[V_x]^t[\boldsymbol{R}][V_x] + 2\mu_{yx}[V_y]^t[\boldsymbol{R}][V_x] \\ & + \mu_{yy}[V_y]^t[\boldsymbol{R}][V_y] \mp 2\mu_{zx}[V_z]^t[\boldsymbol{R}][V_x] \\ & \mp 2\mu_{zy}[V_z]^t[\boldsymbol{R}][V_y] + \mu_{zz}[V_z]^t[\boldsymbol{R}][V_z])\} \end{aligned} \tag{12}$$

and the discretized equivalent of (10) has the form

$$\begin{aligned} F(\bar{V}) = {} & 2\pi\{[V_r]^t(p_{\theta\theta}[D] + m^2 p_{zz}[R] - 2mp_{z\theta}[J])[V_r] \\ & + 2[V_z]^t(\pm mp_{\theta r}[J] + m^2 p_{zr}[R])[V_r] \\ & + [V_z]^t(p_{\theta\theta}[E] + p_{\theta\theta}[N] + \tfrac{1}{4}p_{\theta\theta}[R] \\ & + m^2 p_{rr}[R])[V_z] + 2[V_\theta]^t(mp_{zr}[J]^t - p_{\theta r}[D])[V_r] \\ & \mp [V_\theta]^t(\tfrac{3}{2}p_{z\theta}[R] + 3p_{z\theta}[N] + p_{z\theta}[N]^t + 2p_{z\theta}[E] \\ & + 2mp_{rr}[J]^t)[V_z] + [V_\theta]^t(p_{rr}[D] + p_{zz}[E] \\ & + 3p_{zz}[N] + \tfrac{9}{4}p_{zz}[R])[V_\theta] - \omega^2(\mu_{rr}[V_r]^t[P][V_r] \\ & \mp 2\mu_{zr}[V_z]^t[P][V_r] + \mu_{zz}[V_z]^t[P][V_z] \\ & + 2\mu_{\theta r}[V_\theta]^t[P][V_r] \mp 2\mu_{z\theta}[V_\theta]^t[P][V_z] \\ & + \mu_{\theta\theta}[V_\theta]^t[P][V_\theta])\}. \end{aligned} \tag{13}$$

Equations (12) and (13) can be constructed for a triangle of arbitrary shape provided that the vertex coordinates are known and the constant matrices $[Y_{ij}]$, $[U_{ij}]$, $[\boldsymbol{U}_{ij}]$, $[Q_{ijl}]$, and $[\boldsymbol{Q}_{ijl}]$ defined in the last column of Table I are given.

TABLE II
RESULTS OF THE DISCRETIZATION OF REPRESENTATIVE INTEGRALS FOR FUNCTIONAL (7)

Representative integrals for functional (3.4)	Discretized equivalents in matrix form	Expressions for the square matrices containing the geometrical information about the triangular finite element	Expressions for the constant finite element matrices
$\iint H_1 H_2 \, dxdy$	$[V_1]^t[R][V_2]$	$[R]=2\|A\| \sum_{i=1}^{3} \sum_{j=1}^{3} [Y_{ij}]$	$[Y_{ij}]$ for definition see Table I.
$\iint H_1 \frac{\partial H_2}{\partial y} \, dxdy$	$[V_1]^t[\mathbf{J}][V_2]$	$[\mathbf{J}]=\frac{\|A\|}{2A} \sum_{j=1}^{3} (x_{j+2}-x_{j+1})\{[U_j]+[\mathcal{U}_j]\}$	$[U_j]=\sum_{i=1}^{3} [U_{ij}]$
$\iint H_1 \frac{\partial H_2}{\partial x} \, dxdy$	$[V_1]^t[\mathbf{N}][V_2]$	$[\mathbf{N}]=\frac{\|A\|}{2A} \sum_{j=1}^{3} (y_{j+1}-y_{j+2})\{[U_j]+[\mathcal{U}_j]\}$	$[\mathcal{U}_j]=\sum_{i=1}^{3} [\mathcal{U}_{ij}]$
$\iint \frac{\partial H_1}{\partial y} \frac{\partial H_2}{\partial y} \, dxdy$	$[V_1]^t[\mathbf{D}][V_2]$	$[\mathbf{D}]=\frac{1}{2\|A\|} \sum_{\ell=1}^{3} (x_\ell-x_{\ell+2})(x_\ell-x_{\ell+1})[Q_\ell]$	$[Q_\ell]=\sum_{i=1}^{3} \sum_{j=1}^{3} [Q_{ij\ell}]$
$\iint \frac{\partial H_1}{\partial x} \frac{\partial H_2}{\partial x} \, dxdy$	$[V_1]^t[\mathbf{E}][V_2]$	$[\mathbf{E}]=\frac{1}{2\|A\|} \sum_{\ell=1}^{3} (y_\ell-y_{\ell+2})(y_\ell-y_{\ell+1})[\mathcal{Q}_\ell]$	$[\mathcal{Q}_\ell]=\sum_{i=1}^{3} \sum_{j=1}^{3} [\mathcal{Q}_{ij\ell}]$
$\iint \frac{\partial H_1}{\partial y} \frac{\partial H_2}{\partial x} \, dxdy$	$[V_1]^t[\mathbf{Z}][V_2]$	$[\mathbf{Z}]=\frac{1}{4\|A\|} \sum_{\ell=1}^{3} \{\delta_\ell[Q_\ell]+\tau_\ell[\mathcal{Q}_\ell]\}$ $\delta_\ell=(x_{\ell+1}-x_\ell)(y_\ell-y_{\ell+2})+(x_\ell-x_{\ell+2})(y_{\ell+1}-y_\ell)$ $\tau_\ell=(y_{\ell+1}-y_{\ell+2})(x_\ell-x_{\ell+2})-(y_{\ell+2}-y_\ell)(x_{\ell+2}-x_{\ell+1})$	

NOTE: a) superscript t denotes transposition;
b) subscripts i, j and ℓ are cyclic modulo 3;
c) $[Y_{ij}]$, $[U_j]$ and $[Q_\ell]$ are symmetric, $[\mathcal{U}_j]$ and $[\mathcal{Q}_\ell]$ are antisymmetric matrices;
d) $[Y_{ij}]$, $[U_{ij}]$, $[\mathcal{U}_{ij}]$, $[Q_{ij\ell}]$ and $[\mathcal{Q}_{ij\ell}]$ are defined in Table I;
e) x_i and y_i are triangle vertex coordinates and A is the triangle area.

There are 81 such matrices. However, only 14 are independent, the remaining 67 being obtainable by row and column permutations [15].

V. THE ELEMENT MATRICES

The integrands in the last column of Table I are polynomials in triangle area coordinates. Although straightforward, the integrations [4], [20] are difficult to perform, except in the case $N = 1$ or 2, due to the very large number of algebraic operations required. The author used the IBM PL/I FORMAC compiler to manipulate the polynomial expressions and to perform the integrations symbolically rather than numerically [15]. Thus it was possible to obtain the eight symmetric (Q_{111}, Q_{121}, Q_{221}, Q_{231}, U_{11}, U_{21}, Y_{11}, Y_{21}) [16] and the six antisymmetric ($\boldsymbol{Q}_{111}$, $\boldsymbol{Q}_{112}$, $\boldsymbol{Q}_{121}$, $\boldsymbol{Q}_{231}$, $\boldsymbol{U}_{11}$, $\boldsymbol{U}_{21}$) independent finite element matrices up to and including order 6. The first- and second-order matrices are given in Table III. The higher order matrices ($N = 3$–6) are not reproduced here since they require considerable space. In practice, the numbers are handled by two computer programs which store all 14 independent matrices as integer quotients with common denominators and generate FORTRAN block data statements containing the matrices. One of these programs also performs the summations indicated in the last column of Table II. Thus the element matrices required by (12) are also produced.[2]

[2] For program listing, see NAPS Document Nos. 03004 and 03005 from ASIS/NAPS, c/o Microfiche Publications, P.O. Box 3513, Grand Central Station, New York, NY 10017; remitting \$3.00 per microfiche or \$20.25 per photocopy of Document 03004 and \$21.25 per photocopy of Document 03005.

It should be noted here that the element matrices $[R]/2|A|$ and $[Q_1]$ have originally been given for orders 1–4 by Silvester [4], the antisymmetric matrix $[\boldsymbol{Q}_1]$ has been given independently by Csendes [7], [8] and by Daly [9], [10], and the matrices $[U_1]$ and $[\boldsymbol{U}_1]$ have been computed by Stone [3], [6].

The permutation operations which must be applied to obtain the entire set of 81 element matrices appear in Table IV. There are two basic kinds of permutation operations for element matrices [4], [21]. The first type, denoted by $\boldsymbol{R}$, corresponds to a mapping of the interpolation node numbers by a rotation of the triangle counterclockwise until the last node occupies the relative location of the first one (Fig. 1). The node numbering sequence of the triangle in the standard position corresponds to the row and column sequence of the independent element matrices of Table III. For a second-order triangle in the rotated position, the sequence becomes 6,3,5,1,2,4. The second type of permutation operation, denoted by $\boldsymbol{F}$, corresponds to the mapping of the interpolation node numbers by flipping the triangle about an axis in its plane (Fig. 1). FORTRAN function subroutines implementing the permutation operations $\boldsymbol{R}$ and $\boldsymbol{F}$ and their combinations ($\boldsymbol{R}^2$, $\boldsymbol{RF}$, and $\boldsymbol{R}^2\boldsymbol{F}$) have appeared elsehere [21].

VI. MINIMIZATION OF THE DISCRETIZED FUNCTIONALS

Given the independent element matrices, the matrix forms (12) and (13) can be constructed for any polygonal region that has been subdivided into triangles. In order to assemble a global matrix form, it is required that contribu-

TABLE III
THE FIRST- AND SECOND-ORDER INDEPENDENT ELEMENT MATRICES

First-order matrices

Q_{111} (cd=12)

0		
0	1	
0	−1	1

Q_{121} (cd=24)

0		
0	1	
0	−1	1

Q_{221} (cd=12)

0		
0	1	
0	−1	1

Q_{231} (cd=24)

0		
0	1	
0	−1	1

$\mathcal{Q}_{111}$ (cd=12)

0	−1	0
1	0	0
0	0	0

$\mathcal{Q}_{112}$ (cd=12)

0	0	0
0	0	−1
0	1	0

$\mathcal{Q}_{121}$ (cd=24)

0	−1	0
1	0	0
0	0	0

$\mathcal{Q}_{231}$ (cd=24)

0	−1	0
1	0	0
0	0	0

Y_{11} (cd=360)

12		
3	2	
3	1	2

Y_{21} (cd=360)

3		
2	3	
1	1	1

U_{11} (cd=24)

4		
1	0	
1	0	0

U_{21} (cd=24)

2		
2	0	
1	0	0

$\mathcal{U}_{11}$ (cd=24)

0	1	1
−1	0	0
−1	0	0

$\mathcal{U}_{21}$ (cd=24)

0	2	1
−2	0	0
−1	0	0

First-order matrices

Second-order matrices

Q_{111} (cd=180)

0					
0	96				
0	−96	96			
0	−12	12	7		
0	0	0	−8	16	
0	12	−12	1	−8	7

Q_{121} (cd=360)

0					
0	48				
0	−48	48			
0	8	−8	15		
0	−16	16	−20	32	
0	8	−8	5	−12	7

Q_{221} (cd=180)

0					
0	16				
0	−16	16			
0	12	−12	39		
0	−16	16	−48	64	
0	4	−4	9	−16	7

Q_{231} (cd=360)

0					
0	16				
0	−16	16			
0	4	−4	15		
0	0	0	−16	32	
0	−4	4	1	−16	15

U_{11} (cd=360)

36					
24	64				
24	32	64			
−5	0	−4	0		
4	16	16	0	0	
−5	−4	0	0	0	0

U_{21} (cd=360)

6					
0	96				
0	32	32			
−6	24	0	0		
−8	48	32	0	0	
−1	−8	0	0	0	0

$\mathcal{Q}_{111}$ (cd=180)

0	−60	0	9	−12	0
60	0	24	−4	−8	0
0	−24	0	4	−16	0
−9	4	−4	0	0	0
12	8	16	0	0	0
0	0	0	0	0	0

$\mathcal{Q}_{112}$ (cd=180)

0	0	0	0	0	0
0	0	−96	0	−24	12
0	96	0	−12	24	0
0	0	12	0	−4	1
0	24	−24	4	0	−4
0	−12	0	−1	4	0

$\mathcal{Q}_{121}$ (cd=360)

0	−24	0	1	−4	0
24	0	16	−24	−16	0
0	−16	0	−4	−16	0
−1	24	4	0	0	0
4	16	16	0	0	0
0	0	0	0	0	0

$\mathcal{Q}_{231}$ (cd=360)

0	−4	0	5	8	0
4	0	16	−24	−32	0
0	−16	0	−8	−48	0
−5	24	8	0	0	0
−8	32	48	0	0	0
0	0	0	0	0	0

$\mathcal{U}_{11}$ (cd=360)

0	24	24	−5	4	−5
−24	0	0	0	16	−4
−24	0	0	−4	16	0
5	0	4	0	0	0
−4	−16	−16	0	0	0
5	4	0	0	0	0

$\mathcal{U}_{21}$ (cd=360)

0	16	8	−6	−8	−1
−16	0	0	24	48	−8
−8	0	0	0	32	0
6	−24	0	0	0	0
8	−48	−32	0	0	0
1	8	0	0	0	0

Y_{11} (cd=2520)

24					
12	48				
12	24	48			
−3	−6	−6	2		
0	12	12	−2	8	
−3	−6	−6	1	−2	2

Y_{21} (cd=2520)

3					
0	36				
0	12	12			
−1	0	−2	3		
−2	12	8	0	12	
0	−4	−2	0	−2	1

cd ≡ common denominator

Second-order matrices

tions from each triangle be properly identified and connected together. Since the magnetic field $\bar{H}$ must be continuous from triangle to triangle, the wave function values at adjacent nodes are set equal. Theoretically, the assembly process is performed with connection matrices [5], [22]. However, in practice it is sufficient to employ a global numbering sequence for the interpolation nodes. If the local functionals all constructed using the global node identifiers, the locations of elemental contributions in the global matrix are easily determined. The form of the discretized functionals is unchanged in the global interpretation. For example, the column matrix $[V_r]$ of (13) in global form contains the coefficients from all the triangles in the sequence in which the nodes are numbered. Similarly, in global form, the matrices $[D]$, $[R]$, $[J]$, $[E]$, $[N]$, and $[P]$ are the sum of contributions from the various triangles.

In the global interpretation, (12) and (13) are minimized by differentiating with respect to $[V_r]$, $[V_z]$, $[V_\theta]$, and $[V_x]$, $[V_y]$, $[V_z]$, respectively, and equating the results to zero. For each functional, three matrix equations with three unknown column matrices result; these can be combined into a single matrix equation

$$[S][V] - \omega^2[T][V] = 0. \tag{14}$$

For circulating waves the column matrix $[V]$ and the symmetric coefficient matrices $[T]$ and $[S]$ are given by

$$[V] = \begin{bmatrix} [V_r] \\ [V_z] \\ [V_\theta] \end{bmatrix} \tag{15}$$

$$[T] = \begin{bmatrix} \mu_{rr}[P] & \mp\mu_{zr}[P] & \mu_{\theta r}[P] \\ \mp\mu_{zr}[P] & \mu_{zz}[P] & \mp\mu_{z\theta}[P] \\ \mu_{\theta r}[P] & \mp\mu_{z\theta}[P] & \mu_{\theta\theta}[P] \end{bmatrix} \tag{16}$$

$$[S] = \begin{bmatrix} m^2p_{zz}[R] + p_{\theta\theta}[D] - mp_{z\theta}([J] + [J]^t) & m^2p_{zr}[R] \pm mp_{\theta r}[J]^t & mp_{zr}[J] - p_{\theta r}[D] \\ m^2p_{zr}[R] \pm mp_{\theta r}[J] & p_{\theta\theta}\{\tfrac{1}{4}[R] + \tfrac{1}{2}([N] + [N]^t) + [E]\} + m^2p_{rr}[R] & \mp p_{z\theta}\{\tfrac{3}{4}[R] + \tfrac{3}{2}[N]^t + \tfrac{1}{2}[N] + [E]\} \mp mp_{rr}[J] \\ mp_{zr}[J]^t - p_{\theta r}[D] & \mp p_{z\theta}\{\tfrac{3}{4}[R] + \tfrac{3}{2}[N] + \tfrac{1}{2}[N]^t + [E]\} \mp mp_{rr}[J]^t & p_{zz}\{\tfrac{9}{4}[R] + \tfrac{3}{2}([N] + [N]^t) + [E]\} + p_{rr}[D] \end{bmatrix}. \tag{17}$$

TABLE IV
PERMUTATION OPERATIONS TO OBTAIN ALL 81 SYMMETRIC AND ANTISYMMETRIC ELEMENT MATRICES FROM THE 14 INDEPENDENT MATRICES

Matrices $Q_{ij\ell}$ $\mathcal{Q}_{ij\ell}$	$\ell = 1$			$\ell = 2$			$\ell = 3$		
Y_{ij} U_{ij} $\mathcal{U}_{ij}$	j = 1	j = 2	j = 3	j = 1	j = 2	j = 3	j = 1	j = 2	j = 3
i = 1	Q_{111} given	Q_{121} given	Q_{131} $=FQ_{121}$	Q_{112} $=RFQ_{221}$	Q_{122} $=RFQ_{121}$	Q_{132} $=RQ_{231}$	Q_{113} $=R^2Q_{221}$	Q_{123} $=R^2Q_{231}$	Q_{133} $=R^2Q_{121}$
i = 2	Q_{211} $=Q_{121}$	Q_{221} given	Q_{231} given	Q_{212} $=RFQ_{121}$	Q_{222} $=RQ_{111}$	Q_{232} $=RQ_{121}$	Q_{213} $=R^2Q_{231}$	Q_{223} $=R^2FQ_{221}$	Q_{233} $=R^2FQ_{121}$
i = 3	Q_{311} $=FQ_{121}$	Q_{321} $=Q_{231}$	Q_{331} $=FQ_{221}$	Q_{312} $=RQ_{231}$	Q_{322} $=RQ_{121}$	Q_{332} $=RQ_{221}$	Q_{313} $=R^2Q_{121}$	Q_{323} $=R^2FQ_{121}$	Q_{333} $=R^2Q_{111}$
i = 1	$\mathcal{Q}_{111}$ given	$\mathcal{Q}_{121}$ given	$\mathcal{Q}_{131}$ $=-RF\mathcal{Q}_{231}$	$\mathcal{Q}_{112}$ given	$\mathcal{Q}_{122}$ $=-R^2F\mathcal{Q}_{231}$	$\mathcal{Q}_{132}$ $=R\mathcal{Q}_{231}$	$\mathcal{Q}_{113}$ $=-F\mathcal{Q}_{111}$	$\mathcal{Q}_{123}$ $=R^2\mathcal{Q}_{231}$	$\mathcal{Q}_{133}$ $=R^2\mathcal{Q}_{121}$
i = 2	$\mathcal{Q}_{211}$ $=\mathcal{Q}_{121}$	$\mathcal{Q}_{221}$ $=-RF\mathcal{Q}_{111}$	$\mathcal{Q}_{231}$ given	$\mathcal{Q}_{212}$ $=-R^2F\mathcal{Q}_{231}$	$\mathcal{Q}_{222}$ $=R\mathcal{Q}_{111}$	$\mathcal{Q}_{232}$ $=R\mathcal{Q}_{121}$	$\mathcal{Q}_{213}$ $=R^2\mathcal{Q}_{231}$	$\mathcal{Q}_{223}$ $=R\mathcal{Q}_{112}$	$\mathcal{Q}_{233}$ $=-F\mathcal{Q}_{231}$
i = 3	$\mathcal{Q}_{311}$ $=-RF\mathcal{Q}_{231}$	$\mathcal{Q}_{321}$ $=\mathcal{Q}_{231}$	$\mathcal{Q}_{331}$ $=R^2\mathcal{Q}_{112}$	$\mathcal{Q}_{312}$ $=R\mathcal{Q}_{231}$	$\mathcal{Q}_{322}$ $=R\mathcal{Q}_{121}$	$\mathcal{Q}_{332}$ $=-R^2F\mathcal{Q}_{111}$	$\mathcal{Q}_{313}$ $=R^2\mathcal{Q}_{121}$	$\mathcal{Q}_{323}$ $=-F\mathcal{Q}_{231}$	$\mathcal{Q}_{333}$ $=R^2\mathcal{Q}_{111}$
i = 1	Y_{11} given	Y_{12} $=Y_{21}$	Y_{13} $=Y_{31}$	U_{11} given	U_{12} $=RFU_{21}$	U_{13} $=R^2U_{21}$	$\mathcal{U}_{11}$ given	$\mathcal{U}_{12}$ $=RF\mathcal{U}_{21}$	$\mathcal{U}_{13}$ $=R^2\mathcal{U}_{21}$
i = 2	Y_{21} given	Y_{22} $=RY_{11}$	Y_{23} $=Y_{32}$	U_{21} given	U_{22} $=RU_{11}$	U_{23} $=R^2FU_{21}$	$\mathcal{U}_{21}$ given	$\mathcal{U}_{22}$ $=R\mathcal{U}_{11}$	$\mathcal{U}_{23}$ $=R^2F\mathcal{U}_{221}$
i = 3	Y_{31} $=FY_{21}$	Y_{32} $=RY_{21}$	Y_{33} $=R^2Y_{11}$	U_{31} $=FU_{21}$	U_{32} $=RU_{21}$	U_{33} $=R^2U_{11}$	$\mathcal{U}_{31}$ $=F\mathcal{U}_{21}$	$\mathcal{U}_{32}$ $=R\mathcal{U}_{21}$	$\mathcal{U}_{33}$ $=R^2\mathcal{U}_{11}$

Note: $\bar{R}$ denotes the rotation permutation operator, F denotes the flip permutation operator, and $R^2 \equiv RR$.

For traveling waves, $[V]$, $[T]$, and $[S]$ become

$$[V] = \begin{bmatrix} [V_x] \\ [V_y] \\ [V_z] \end{bmatrix} \tag{18}$$

$$[T] = \begin{bmatrix} \mu_{xx}[R] & \mu_{yx}[R] & \mp\mu_{zx}[R] \\ \mu_{yx}[R] & \mu_{yy}[R] & \mp\mu_{zy}[R] \\ \mp\mu_{zx}[R] & \mp\mu_{zy}[R] & \mu_{zz}[R] \end{bmatrix} \tag{19}$$

$$[S] = \begin{bmatrix} \beta^2 p_{yy}[R] + p_{zz}[D] + \beta p_{zy}([J] + [J]^t) & -\beta^2 p_{yx}[R] - \beta p_{zx}[J]^t - p_{zz}[Z] - \beta p_{zy}[N] & \mp\beta p_{yx}[J] \mp p_{zx}[D] \pm \beta p_{yy}[N] \pm p_{zy}[Z] \\ -\beta^2 p_{yx}[R] - \beta p_{zx}[J] - p_{zz}[Z]^t - \beta p_{zy}[N]^t & \beta p_{zx}([N] + [N]^t) + p_{zz}[E] + \beta^2 p_{xx}[R] & \mp\beta p_{yx}[N] \pm p_{zx}[Z]^t \pm \beta p_{xx}[J] \mp p_{zy}[E] \\ \mp\beta p_{yx}[J]^t \mp p_{zx}[D] \pm \beta p_{yy}[N]^t \pm p_{zy}[Z]^t & \mp\beta p_{yx}[N]^t \pm p_{zx}[Z] \pm \beta p_{xx}[J]^t \mp p_{zy}[E] & -p_{yx}([Z] + [Z]^t) + p_{yy}[E] + p_{xx}[D] \end{bmatrix}. \tag{20}$$

For an assembly of triangles with a total of n_t interpolation nodes, the size of the coefficient matrices $[S]$ and $[T]$ is $3n_t \times 3n_t$. The eigenvalues of (14) are ω^2 and the eigenvectors are the nodal values of the vector fields. For every value of the propagation constant β, or of the circulation constant m, an eigenvalue–eigenvector spectrum set can be obtained. To obtain the magnetic field for circulating waves, the solutions of (14) must be inversely transformed in accordance with (9).

VII. ACCURACY OF THE METHOD

Consider a rectangular waveguide with a 2:1 width-to-height ratio completely filled with a ferrite material characterized by the relative permeability tensor

$$\hat{\mu}_r = \begin{bmatrix} 3.0 & 0.0 & +j0.8 \\ 0.0 & 1.0 & +j0.0 \\ -j0.8 & -j0.0 & 3.0 \end{bmatrix} \tag{21}$$

and a relative permittivity of 2. Here $\hat{\mu}_r$ is independent of frequency, although this assumption is not valid for ferrites in general [11], [12]. An analytical solution can be obtained for the dominant waveguide mode and also for some of the higher order modes [15]. A finite element solution was obtained with two sixth-order triangles. The comparison of the finite element solution with the analytical solution appears in Tables V and VI.

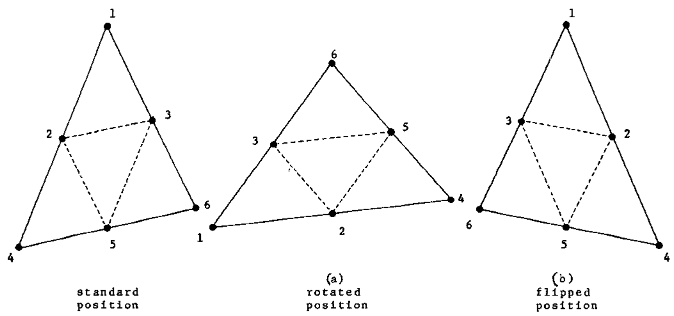

Fig. 1. Mapping of the interpolation nodes of a second-order triangle by (a) counterclockwise rotation, and (b) flip over about an axis in its plane.

TABLE V
THE WAVENUMBERS k_1, k_2, AND k_3 AND THE PERCENT ERRORS FOR THE FERRITE-FILLED RECTANGULAR WAVEGUIDE SOLVED BY THE FINITE ELEMENT METHOD USING TWO SIXTH-ORDER TRIANGLES

β	k_1	% error*	k_2	% error*	k_3	% error*
-1	0.788758	1.74×10^{-4}	1.39655	1.64×10^{-3}	2.06006	0.956
0	0.665367	2.56×10^{-4}	1.33076	1.70×10^{-3}	2.01606	0.999
+1	0.788758	1.74×10^{-4}	1.39655	1.64×10^{-3}	2.06006	0.956
+2	1.07723	2.42×10^{-4}	1.57754	1.35×10^{-3}	2.18662	0.838
+3	1.43441	3.89×10^{-4}	1.84005	1.32×10^{-3}	2.38326	0.717
+4	1.82031	0.06×10^{-4}	2.15448	1.17×10^{-3}	2.62255	0.164
+5	2.21999	0.59×10^{-4}	2.50133	1.15×10^{-3}	2.92135	0.378

*analytical solution: $k_n^2 = \frac{3}{16.72}[\beta^2+(n\frac{\pi}{2})^2]$, (n=1,2,3,...), $k_n=\omega_n\sqrt{\mu_o\epsilon_o}$

TABLE VI
VALUES OF H_z AT THE INTERPOLATION NODES AT $y = 0$ AND PERCENT ERRORS FOR THE FERRITE-FILLED RECTANGULAR WAVEGUIDE SOLVED BY THE FINITE ELEMENT METHOD USING TWO SIXTH-ORDER TRIANGLES

β	$H_z(x=-1)$ / % error*	$H_z(x=-\frac{2}{3})$ / % error*	$H_z(x=-\frac{1}{3})$ / % error*	$H_z(x=0)$ / % error*	$H_z(x=+\frac{1}{3})$ / % error*	$H_z(x=+\frac{2}{3})$ / % error*	$H_z(x=+1)$ / % error*
-1	-4.3323	-4.1199	-2.8028	-0.73543	+1.5289	+3.3845	+4.3323
	.000	-.007	.010	.006	.020	-.010	.000
0	-5.1357	-4.4480	-2.5674	+0.00006	+2.5676	+4.4481	+5.1358
	.002	-.006	.019	----	.012	-.008	.000
+1	-4.3323	-3.3845	-1.5289	+0.73546	+2.8028	+4.1199	+4.3323
	.000	-.010	.020	.002	.010	-.007	.000
+2	-3.1721	-2.2089	-0.65313	+1.0771	+2.5187	+3.2859	+3.1721
	.000	-.013	.029	-.007	.003	-.008	.000
+3	-2.3821	-1.4568	-0.14028	+1.2133	+2.2418	+2.6700	+2.3821
	.000	-.030	.081	-.009	.001	-.017	.000
+4	-1.8767	-0.98929	+0.16511	+1.2757	+2.0424	+2.2640	+1.8775
	.043	-.080	.162	-.060	.023	-.025	.000
+5	-1.5391	-0.67965	+0.36195	+1.3064	+1.9012	+1.9862	+1.5389
	-.013	-.008	-.040	-.011	-.026	-.017	.000

*analytical solution: $H_z=A[\sin(\frac{\pi}{2}x)+(1.6\beta/3\pi)\cos(\frac{\pi}{2}x)]$ where A has been chosen in such a way that the percent error in the values $H_z(x=+1)$ computed by the finite element method is zero.

VIII. CONCLUSIONS

The three-component vector variational formulation coupled with the high-order polynomial triangular finite element method presented in this paper has the following significant features and advantages.

1) The formulation is based on specific wave forms such as traveling waves which are linearly polarized in the transverse plane and circulating waves which are circularly polarized in the r-z plane. Depending on the choice of material property tensor forms, other types of waves (e.g., circulating waves which are linearly polarized in the r-z plane) could be treated in a similar way.

2) The formulation is valid for electromagnetic waves both in isotropic and anisotropic lossless media with the resulting coefficient matrices being real in both cases.

3) For inhomogeneous media, no singularities will appear at the air line and dielectric line on the dispersion diagram.

4) The finite elements are assembled from precalculated constant matrices. Thus the matrix assembly does not involve integration and consequently the assembly process is fast and relatively easy to carry out.

5) Waveguides of arbitrary cross section or cavities of arbitrary longitudinal cross section can be handled, and provided that the finite element model has sufficient degrees of freedom, the results are very accurate.

The following are the limitations and drawbacks of the method.

1) The material property tensors are assumed to be independent of the frequency and they must not have spatial variation within the triangles.

2) It is difficult to model curved boundaries with triangles which have straight edges.

3) Since three vector components must be considered for every interpolation node, the size of the global matrix equation is larger than in two-component vector formulations. Explicit enforcement of boundary conditions such as $\bar{n}\cdot\bar{B} = 0$ or $\bar{n}\cdot\bar{D} = 0$ is advantageous for reducing the matrix size and for eliminating nonphysical solutions from the spectrum as described in [1].

Although the derivations in this paper are given in terms of the magnetic field, the results are also valid for the electric field provided that dual quantities are substituted and the appropriate boundary conditions are added.

ACKNOWLEDGMENT

The author will be forever grateful to Prof. Silvester for the generous assistance received and the invaluable knowledge gained from him during the many years of association with him.

REFERENCES

[1] A. Konrad, "Vector variational formulation of electromagnetic fields in anisotropic media," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, pp. 553–559, Sept. 1976.
[2] W. J. English and F. J. Young, "An *E* vector variational formulation of the Maxwell equations for cylindrical waveguide problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 40–46, Jan. 1971.
[3] G. O. Stone, "High-order finite elements for inhomogeneous acoustic guiding structures," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-21, pp. 538–542, Aug. 1973.
[4] P. Silvester, "High-order polynomial triangular finite elements for potential problems," *Int. J. Eng. Sci.*, vol. 7, pp. 849–861, 1969.
[5] ——, "A general high-order finite element waveguide analysis program," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 204–210, Apr. 1969.
[6] G. O. Stone, "Coupling matrices for high-order finite element analysis of acoustic-wave propagation," *Electron. Lett.*, vol. 8, pp. 466–468, Sept. 7, 1972.
[7] Z. J. Csendes and P. Silvester, "Numerical solution of dielectric loaded waveguides: I—Finite-element analysis," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 1124–1131, Dec. 1970.
[8] Z. J. Csendes, "Solution of dielectric loaded waveguides by finite element methods," M.Eng. thesis, McGill Univ., Mar. 1970.
[9] P. Daly, "Hybrid-mode analysis of microstrip by finite-element methods," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 19–25, Jan. 1971.
[10] ——, "Finite-element coupling matrices," *Electron. Lett.*, vol. 5, pp. 613–615, Nov. 27, 1969.
[11] B. Lax and K. J. Button, *Microwave Ferrites and Ferrimagnetics.* New York: McGraw-Hill, 1962.
[12] S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics.* New York: Wiley, 1967.
[13] F. E. Gardiol, "Computer analysis of latching phase shifters in rectangular waveguide," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-21, pp. 57–61, Jan. 1973.
[14] S. F. Borg, *Matrix-Tensor Methods in Continuum Mechanics.* Princeton, NJ: D. Van Nostrand, 1963.
[15] A. Konrad, "Triangular finite elements for vector fields in electromagnetics," Ph.D. thesis, McGill Univ., Sept. 1974 (Thesis TC-24352, available from: National Library, Canadian Theses Division, 395 Wellington St., Ottawa, Ont., Canada, K1A 0N4.)
[16] A. Konrad and P. Silvester, "Triangular finite elements for the generalized Bessel equation of order *m*," *Int. J. Numer. Meth. Eng.*, vol. 7, pp. 43–55, 1973.
[17] A. Konrad, "Linear accelerator cavity field calculation by the finite element method," *IEEE Trans. Nucl. Sci.*, vol. NS-20, pp. 802–808, Feb. 1973.
[18] A. Konrad and P. Silvester, "A finite element program package for axisymmetric vector field problems," *Computer Physics Communications*, vol. 9, pp. 193–204, 1975.
[19] O. C. Zienkiewicz, *The Finite Element Method in Engineering Science.* New York: McGraw-Hill, 1971.
[20] G. C. Best, "Helpful formulas for integrating polynomials in three dimensions," *Math. Comp.*, vol. 18, pp. 310–312, 1964.
[21] P. Silvester and A. Konrad, "Axisymmetric triangular finite elements for the scalar Helmholtz equation," *Int. J. Numer. Meth. Eng.*, vol. 5, pp. 481–497, 1973.
[22] J. T. Oden, *Finite Elements of Nonlinear Continua.* New York: McGraw-Hill, 1972.

High-Azimuthal-Index Resonances in Ferrite MIC Disk Resonators

PIETRO DE SANTIS, MEMBER, IEEE

***Abstract*—This paper presents a study of the nonreciprocal high-azimuthal-index zero-radial-order modes which may resonate in ferrite MIC disk resonators magnetized perpendicularly to the ground plane. Both ferrite volume (FV) and edge-guided-wave (EGW) modes are investigated by using a suitable equivalent model. It is found that when the ferrite is saturated, a simple empirical parameter is sufficient to characterize the fringing-field effects at the disk's edge.**

I. INTRODUCTION

RECENTLY [1], [2], ferrite MIC disk resonators of large diameter have received some attention because they are suitable to study the propagation characteristics of the "edge-guided" waves (EGW) [3] in very much the same way as MIC ring resonators were used to study quasi-TEM propagation in isotropic MIC's.

Manuscript received May 28, 1976; revised November 5, 1976.
The author is with the Naval Research Laboratory, Washington, DC 20375, on leave from Selenia S.p.A., Rome, Italy, and the University of Naples, Naples, Italy.

Among the various modes which may resonate in such disk resonators, those appropriate to studying EGW characteristics are the $TM_{0,n,0}$ modes with $n = 4,5,6,\cdots$. These modes are TM with respect to the Z-axis, which is taken perpendicular to the ground plane. They present no nodes in the radial direction and are Z-independent.

An approximate analysis of a ferrite disk resonator magnetized perpendicularly to the ground plane was developed by the present author [4] using perfect magnetic-wall boundary conditions at the disk's edge. Subsequent experiments carried out by Brundle [2] measured EGW phase velocities 8 percent off the theoretical predictions.

The disagreement between theory and experiment was probably due to the use of unaccurate boundary conditions; i.e., to the neglect of fringing-field effects.

It is the purpose of this paper to study the fringing-field effects in ferrite MIC disk resonators and, more specifically, to evaluate how they affect the $TM_{0,n,0}$ resonances. The